

**Modified Enlarged 18pt**

**OXFORD CAMBRIDGE AND RSA EXAMINATIONS**

**Friday 12 June 2020 – Afternoon**

**A Level Mathematics B (MEI)**

**H640/03 Pure Mathematics and Comprehension**

**Insert**

**Time allowed: 2 hours  
plus your additional time allowance**

**READ INSTRUCTIONS OVERLEAF**



## **INSTRUCTIONS**

**Do NOT send this Insert for marking. Keep it in the centre or recycle it.**

## **INFORMATION**

**This Insert contains the article for Section B.**

## Which is bigger?

Which is bigger:  $\pi^e$  or  $e^\pi$ ? Using a calculator confirms that  $e^\pi$  is the larger, but how can this be proved without the use of a calculator?

### Simpler examples

It is often helpful in mathematics to consider simpler examples. It is easy to work out that  $3^4 > 4^3$ . In the expression  $3^4$ , 3 is the base and 4 is the exponent. Working with integers greater than 1, it is easy to find many examples where  $a^b > b^a$  if  $a < b$ . That is, using the smaller base and the larger exponent gives the larger result. This might lead us to conjecture that  $a^b > b^a$  if  $a < b$  and both  $a$  and  $b$  are integers greater than 1. However, it is also possible to find counter examples to this conjecture. 5 10

Exponents can also be rational numbers, and in general  $x^{\frac{p}{q}}$  denotes  $(\sqrt[q]{x})^p$  where  $p$  and  $q$  are integers and  $q$  is positive. So, any rational power of a positive number,  $x$ , can be defined. However, both  $e$  and  $\pi$  are irrational numbers. Considering the original question about  $\pi^e$  and  $e^\pi$  raises the issue of what is meant by an irrational power of a number. 15 20

### Extending the definition of power to irrational numbers

What, for example, is meant by  $2^\pi$ ?

An irrational number corresponds to a non-recurring infinite decimal. Rounding the decimal gives a rational approximation to the irrational number. For example, the following sequence gives increasingly accurate approximations to  $\pi$ . 25

3, 3.1, 3.14, 3.142, 3.1416, 3.14159, ... 30

Using a spreadsheet gives a sequence of approximations to  $2^\pi$ , as shown in Fig. C1. The limit of this sequence of approximations is the value of  $2^\pi$ . This limit cannot be evaluated with a spreadsheet but it is, in principle, possible to find the value to any required degree of accuracy. 35

FIG. C1

	A	B	
1	$k$	$2^k$	
2	3	8	
3	3.1	8.574188	
4	3.14	8.815241	
5	3.142	8.82747	
6	3.1416	8.825023	
7	3.14159	8.824962	

$2^x$  and  $x^2$  are increasing functions of  $x$  for  $x > 0$  and this allows us to deduce that  $\pi^2 > 2^\pi$ , as follows.

**We know that  $\pi$  is between 3 and 3.142**

$$\pi < 3.142 \Rightarrow 2^\pi < 2^{3.142} = 8.827\,47$$

**40**

$$\pi > 3 \Rightarrow \pi^2 > 3^2 = 9$$

$$\text{So } \pi^2 > 9 > 8.827\,47 > 2^\pi$$

$$\text{Hence } \pi^2 > 2^\pi$$

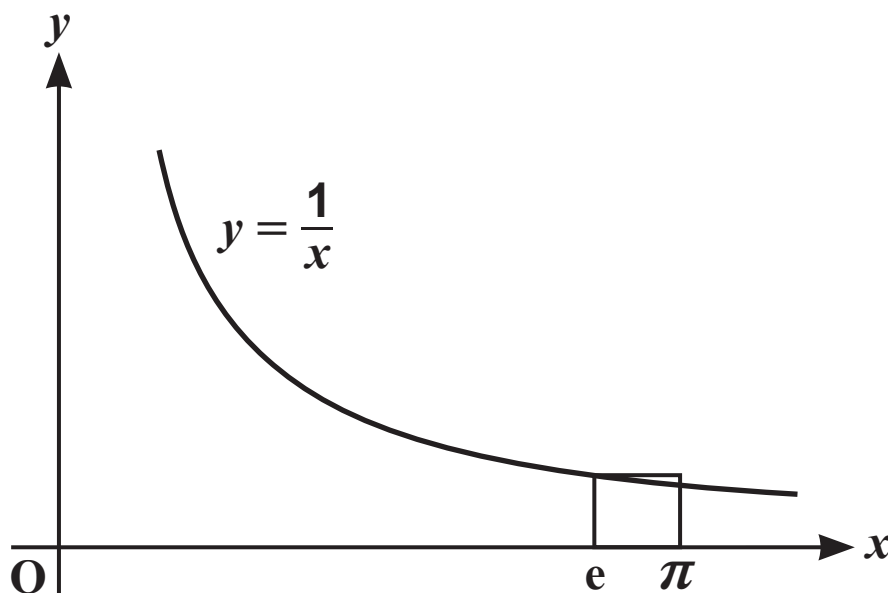
Which is bigger:  $\pi^e$  or  $e^\pi$ ?

An indirect method, using calculus, enables us to prove that  $e^\pi$  is larger than  $\pi^e$ . Fig. C2 shows the curve  $y = \frac{1}{x}$  in the first quadrant together with the rectangle with vertices at the points  $(e, 0)$ ,  $(e, \frac{1}{e})$ ,  $(\pi, \frac{1}{e})$  and  $(\pi, 0)$ . We use the fact that the area under the curve between  $e$  and  $\pi$  is less than the area of this rectangle.

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FIG. C2



The area of the rectangle is  $\frac{1}{e}(\pi - e)$

$$\int_e^\pi \frac{1}{x} dx < \frac{1}{e}(\pi - e)$$

$$\ln \pi - 1 < \frac{\pi}{e} - 1$$

$$\ln \pi < \frac{\pi}{e}$$

55

$e^x$  is an increasing function for all values of  $x$

hence  $\pi < e^{\frac{\pi}{e}}$

Assuming that the usual rules of indices apply to irrational powers of irrational numbers, raising both sides of the inequality to the power  $e$  gives the desired result. 60

Using a similar method, it can be shown that  $e^a > a^e$  for any positive number  $a \neq e$ .

An alternative method for showing that  $e^a > a^e$  for any positive number  $a$  is to show that the only stationary point on the curve  $y = \frac{\ln x}{x}$  (a maximum) occurs where  $x = e$ . 65



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